Möbius Transformations

Contents

1 Introduction

Möbius transformations, named in honour of German mathematician August Ferdinand Möbius (1790-1868), appear in many complex analysis courses, but are often not discussed in depth. However, these transformations are well worth further study, as they posses many interesting and useful properties, and furthermore, a satisfying and complete classification is possible.

Moreover, aside from their main property of being exactly the bijective, conformal maps of the Riemann sphere, they crop up seemingly unexpectedly in many areas of mathematics; such as projective geometry, group theory, and even general relatively.

2 The Extended Complex Plane and the Riemann Sphere

When we consider Möbius transformations, it is more convenient to consider the *extended complex place*, $\hat{\mathbb{C}}$, defined by $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. How do we treat this point ∞? We use the following common conventions, as in [4, pp. 203]

$$a \pm \infty = \pm\infty + a = \infty \qquad a/\infty = 0 \qquad \forall a \in \mathbb{C}$$
$$a \times \infty = \infty \times a = \infty \qquad a/0 = \infty \qquad \forall a \in \mathbb{C} \setminus \{0\}$$

These algebraic definitions for this point make intuitive sense: we are used to the fact that $\forall a \in \mathbb{C}$, $\lim_{|z|\to\infty} \frac{a}{z} = 0$, and $\forall a \in \mathbb{C} \setminus \{0\}$, $\lim_{|z|\to 0} \frac{a}{z} = \infty$. Note also that $\frac{0}{0}$ and $\frac{\infty}{\infty}$ are still undefined.

We can give this geometric meaning, by first thinking of $\mathbb{C}$ as being embedded inside of $\mathbb{R}^3$, with $z = x + iy$ identified with the point $(x, y, 0)$, so $\mathbb{C}$ is the plane $\{(x, y, z) \in \mathbb{R}^3 \mid z = 0\}$.

The *Riemann Sphere*, $\mathbf{\Sigma}$, in $\mathbb{R}^3$, is the sphere which has the unit circle as it's intersection with the xy-plane: $\mathbf{\Sigma} = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$.

Set $N = (0, 0, 1)$ - the north pole. Then we can define the following projection ψ from the Sphere excluding the north pole, onto the complex plane. Geometrically, for each point P on the sphere, we draw the line through N and P and assigns $\psi(P)$ as the intersection of this line with the xy-plane.

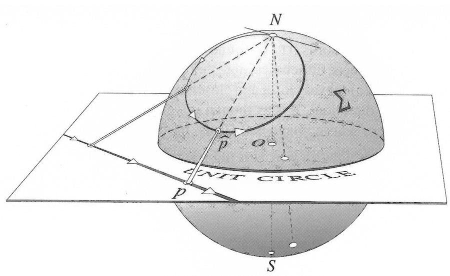

Figure 1: The Stereographic Projection

Definition 2.1. The *Stereographic Projection* $\psi : \mathbf{\Sigma} \setminus \{N\} \to \mathbb{C}$ is defined as:

$$\psi : (x, y, z) \mapsto \left(\frac{x}{1-z}\right) + \left(\frac{y}{1-z}\right) i$$

We can further extend this to $\psi : \mathbf{\Sigma} \to \hat{\mathbb{C}}$ by setting $\psi(N) = \infty$. This assignment makes intuitive sense; using the geometrical viewpoint above, "the line through N and N" is not uniquely defined, but as P moves towards N, the line through N and P approaches the horizontal and so the intersection with the xy-plane gets further and further away from the origin. For more information about ψ see [6, pp. 140].

We will not prove this, but it doesn't take long to show that ψ is a bijection between $\mathbf{\Sigma}$ and $\hat{\mathbb{C}}$. Furthermore, under this correspondence, circles on the sphere not passing through N are circles in the plane, and the circles that pass through N correspond to lines in the plane in addition to the point at infinity. Therefore, if we are viewing $\hat{\mathbb{C}}$ as the sphere we can remove the distinction and talk about *generalised circles* or *circles in* $\hat{\mathbb{C}}$ defined as: either a Euclidean circle in $\mathbb{C}$ or the union of a Euclidean line L in $\mathbb{C}$, with ∞, which we denote by $\bar{L} = L \cup \{\infty\}$.

For example, the extended real axis, $\bar{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ is the generalised circle in $\hat{\mathbb{C}}$ containing the real axis. On the Riemann sphere this corresponds to the great circle through $N, (0, 0, -1), (1, 0, 0)$.

3 Möbius Transformations

Now with the preliminaries dealt with, we can define the transformations themselves.

Definition 3.1. [4, pp. 203] A *Möbius Transformation*[1] is map $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ of the form

$$f(z) = \frac{az+b}{cz+d} \quad \text{where} \quad a,b,c,d \in \mathbb{C},\ ad-bc \neq 0$$

The requirement for $ad - bc \neq 0$ is to ensure all such transformations are invertible, as we shall soon see. By computing f in the case $ad - bc = 0$ we see that f is then the trivial map to a single point, so is definitely not invertible.

Notice, that if f is a Möbius transformation, then

$$f(z) = \frac{az+b}{cz+d} = \frac{(\lambda a)z + \lambda b}{(\lambda c)z + \lambda d} \quad \forall \lambda \in \mathbb{C} \setminus \{0\}$$

Therefore, the representation is not unique. However we can achieve near uniqueness by assuming $ad - bc = 1$, because we can always 'normalise' by dividing each coefficient by $\sqrt{ad-bc}$. Consequently, we define the set of Möbius transformations as

$$M\ddot{o}b(\hat{\mathbb{C}}) = \{f : \hat{\mathbb{C}} \to \hat{\mathbb{C}} \,|\, f \text{ is a Möbius transformation, and } ad - bc = 1\}$$

By near uniqueness, we mean that the representation in this set is unique up to a factor of -1. I.e. both $f(z) = \frac{az+b}{cz+d}$, $g(z) = \frac{(-a)z+(-b)}{(-c)z+(-d)}$ are the same map, and these are **all** the representatives of this map.

3.1 Correspondence with Matrix groups

The first key observation is that this set forms a group.

Proposition 3.2. *$M\ddot{o}b(\hat{\mathbb{C}})$ forms a group under composition of functions.*

Proof. The identity element, $id \in M\ddot{o}b(\hat{\mathbb{C}})$ (take $a = d = 1$, $b = c = 0$).

Take $f, g \in M\ddot{o}b(\hat{\mathbb{C}})$, $f : z \mapsto \frac{az+b}{cz+d}$, $g : z \mapsto \frac{ez+f}{gz+h}$, then:

$$f \circ g : z \mapsto \frac{a\frac{ez+f}{gz+h} + b}{c\frac{ez+f}{gz+h} + d} = \frac{z(ae+bg) + (af+bh)}{z(ce+dg) + (cf+dh)}$$

$f \circ g$ is certainly in the correct form of a Möbius transformation. Additionally, as $f, g \in M\ddot{o}b(\hat{\mathbb{C}})$, then $(ae+bg)(cf+dh) - (ce+dg)(af+bh) = (ad-bc)(eh-fg) = 1$, hence $f \circ g \in M\ddot{o}b(\hat{\mathbb{C}})$, so $M\ddot{o}b(\hat{\mathbb{C}})$ is closed under composition.

Furthermore, $h : z \mapsto \frac{dz-b}{-cz+a}$ satisfies $da - (-b)(-c) = ad - bc = 1$ and $f \circ h = f \circ h = id$ so $h = f^{-1} \in M\ddot{o}b(\hat{\mathbb{C}})$.

Therefore, as we know that composition of functions is associative, $M\ddot{o}b(\hat{\mathbb{C}})$ indeed forms a group.

□

From this we can see elements of $M\ddot{o}b(\hat{\mathbb{C}})$ are bijective, as they posses both left and right inverses.

You may have noticed that the form of the composition and inverse of a Möbius transformations looks suspiciously like the product and inverse of 2 by 2 matrices. Let's make this correspondence formal.

Theorem 3.3. *$\phi : SL(2,\mathbb{C}) \to M\ddot{o}b(\hat{\mathbb{C}})$ defined by:*

$$\phi : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto f : z \mapsto \frac{az+b}{cz+d}$$

is a group homomorphism.

[1] Also commonly referred to as *Fractional Linear Transformations* or *Bilinear Transformations*.

Proof. Take $A, B \in SL(2,\mathbb{C})$, $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $B = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$.

Firstly, this is well defined as $\det(A) = 1$ ensures that $\phi(A) \in M\ddot{o}b(\hat{\mathbb{C}})$.

Now let $f := \phi(A)$, $g := \phi(B)$. Note that:

$$f \circ g : z \mapsto \frac{a\frac{ez+f}{gz+h} + b}{c\frac{ez+f}{gz+h} + d} = \frac{z(ae+bg) + (af+bh)}{z(ce+dg) + (cf+dh)}$$

Therefore:

$$\begin{aligned} \phi(AB) &= \phi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e & f \\ g & h \end{pmatrix}\right) \\ &= \phi\left(\begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix}\right) \\ &= f \circ g \\ &= \phi(A) \circ \phi(B) \end{aligned}$$

□

Corollary 3.4. $M\ddot{o}b(\hat{\mathbb{C}}) \cong SL(2,\mathbb{C}) \big/ \{I_2, -I_2\}$

Proof. If we can show $ker(\phi) = \{I_2, -I_2\}$, then by the first isomorphism theorem the result follows, as ϕ is clearly surjective.

Recall $ker(\phi) = \{A \in SL(2,\mathbb{C}) \,|\, \phi(A) = id\}$. So if $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in ker(\phi)$, then $b = c = 0$, $\frac{a}{d} = 1$, which means $a = d$. Also $\det(A) = ad - bc = ad = 1$ so either $a = d = 1$ or $a = d = -1$. Hence $ker(\phi) = \{I_2, -I_2\}$ □

We denote $SL(2,\mathbb{C}) \big/ \{I_2, -I_2\}$ by $PSL(2,\mathbb{C})$ which stands for projective special linear group[2].

This homomorphism determines the equivalence relation $A \sim B \Leftrightarrow \phi(A) = \phi(B)$. So A and B are in the same equivalence class if and only if the are the same up to sign. We write $[A] = \{A, -A\}$ as the equivalence class of A, so for any $f \in M\ddot{o}b(\hat{\mathbb{C}})$, $f \leftrightarrow [A]$, where A is such that $\phi(A) = f$. The operation in $PSL(2,\mathbb{C})$ is $[A][B] = [AB]$.

Henceforth, when we talk about $f \in M\ddot{o}b(\hat{\mathbb{C}})$ keep closely in your mind the corresponding element $[A] \in PSL(2,\mathbb{C})$. This alternative viewpoint will be very useful later on.

3.2 Elementary Properties

At first sight, it does not seem clear what the action of these transformations on $\hat{\mathbb{C}}$ looks like. However, the following helps visualisation and understanding tremendously.

Proposition 3.5. *Any Möbius transformation $F : z \mapsto \frac{az+b}{cz+b}$ can be built up from maps of the following types:*

(T1)	$z \mapsto az$	$(b = c = 0, d = 1)$
(T2)	$z \mapsto z + b$	$(c = 0, a = d = 1)$
(T3)	$z \mapsto 1/z$	$(a = d = 0, b = c = 1)$

Proof. Taken from [4, pp. 205]

Case 1. $c = 0$

By the restriction that $ad - bc \neq 0$, $d \neq 0$, so

$$F = f_1 \circ f_2, \text{ where } f_1 : z \mapsto z + \frac{b}{d},\ f_2 : z \mapsto \frac{a}{d}z$$

Case 2. $c \neq 0$ Let

$$f_1 : z \mapsto z + \frac{a}{c}, \quad f_2 : z \mapsto \frac{1}{c}(bc - ad)z, \quad f_3 : z \mapsto \frac{1}{z}, \quad f_4 : z \mapsto z + d, \quad f_5 : z \mapsto cz$$

Then $F = f_1 \circ f_2 \circ f_3 \circ f_4 \circ f_5$. □

[2]Why will call it this will become evident later on.

Example 3.6. Decompose the transformation $f(z) = \frac{iz}{z-i}$ into maps of the above types.

As $c \neq 0$, using the above proof, $f = f_1 \circ f_2 \circ f_3 \circ f_4$, where

$$f_1 : z \mapsto z + i, \quad f_2 : z \mapsto -z, \quad f_3 : z \mapsto \frac{1}{z}, \quad f_4 : z \mapsto z - i$$

So f is just a translation by $-i$, followed by an inversion, then a rotation by π, and finally a translation by i.

We can write $a = Re^{i\theta}$, so (T1) can be interpreted geometrically as a rotation by θ anticlockwise, followed by an enlargement about 0 by a scale factor of R. (T2) is a translation by b. (T3), however, is more complicated and we refer to it as the inversion map.

If we write $z = re^{i\theta}$, then the image $w := \frac{1}{z} = \frac{1}{r}e^{-i\theta}$. So points outside the unit disk are mapped to the point of the same argument inside the unit disk, then reflected in the real axis, and vice versa. In particular $0 \mapsto \infty$ and $\infty \mapsto 0$.

Recall the stereographic projection, under which points on the lower hemisphere get mapped to inside the unit disk and points on the upper hemisphere go to outside the unit disk. With this similarity as motivation, let us consider $(\psi \circ T \circ \psi^{-1})(z)$ where $T : \mathbb{R}^3 \mapsto \mathbb{R}^3$, $T(x, y, z) = (x, -y, -z)$. Geometrically T is a rotation about the x axis by π. Using the inverse formula for ψ (which you can easily verify), if $z = a + ib$:

$$\begin{aligned}
(\psi \circ T \circ \psi^{-1})(z) &= (\psi \circ T \circ \psi^{-1})(a + ib) \\
&= \psi(T(\psi^{-1}(a + ib))) \\
&= \psi\left(T\left(\frac{1}{1 + a^2 + b^2}(2a, 2b, -1 + a^2 + b^2)\right)\right) \\
&= \psi\left(\frac{1}{1 + a^2 + b^2}(2a, -2b, 1 - a^2 - b^2)\right) \\
&= \frac{2a - 2ib}{2a^2 + 2b^2} = \frac{1}{a + ib} = \frac{1}{z}
\end{aligned}$$

Therefore, $\psi \circ T \circ \psi^{-1}$ is exactly (T3), so we can view an inversion instead much more simply as just a rotation of there sphere. We will return to this idea in a much more powerful way in section 5.

One useful property that is exploited in hyperbolic geometry, to map the Cayley-Klein disk to Poincaré half-plane is the following:

Proposition 3.7. *For any distinct $z_1, z_2, z_3 \in \hat{\mathbb{C}}$, distinct $w_1, w_2, w_3 \in \hat{\mathbb{C}}$, there exists a unique Möbius transformation taking $z_1 \mapsto w_1$, $z_2 \mapsto w_2$, $z_3 \mapsto w_3$.*

Proof. Taken from [4, pp. 206] Consider both

$$g : z \mapsto \left(\frac{z - z_1}{z - z_3}\right)\left(\frac{z_2 - z_3}{z_2 - z_1}\right), \qquad h : w \mapsto \left(\frac{w - w_1}{w - w_3}\right)\left(\frac{w_2 - w_3}{w_2 - w_1}\right)$$

As $\{z_1, z_2, z_3\}$ and $\{w_1, w_2, w_3\}$ are distinct then these are both are in the correct form of a Möbius transformation ($ad - bc \neq 0$). g (as you can easily verify) sends z_1, z_2, z_3 to $0, 1, \infty$ respectively, and h sends w_1, w_2, w_3 to $0, 1, \infty$. Therefore, $h^{-1} \circ g$ takes z_1, z_2, z_3 to w_1, w_2, w_3. As $M\ddot{o}b(\hat{\mathbb{C}})$ is a group, the composition and inverse of Möbius transformations are themselves Möbius transformations, so we have proven existence - all that is left to do is show that this is unique.

Suppose that $m : z \mapsto (az + b)/(cz + d)$ has fixed points of $0, 1, \infty$, i.e. $m(0) = 0$, $m(1) = 1$, $m(\infty) = \infty$. $m(0) = 0 \Rightarrow b = 0$, $\quad m(\infty) = \infty \Rightarrow c = 0$, hence $m(1) = 1 \Rightarrow a = d$. Therefore, $m = id$.

In light of this, take p, a Möbius transformation that also sends each $z_i \mapsto w_i$. Then $h \circ p \circ g^{-1}$ fixes $0, 1, \infty$, so by the above, $h \circ p \circ g^{-1} = id$, hence $p = h^{-1} \circ g$. So the transformation is unique.

□

Example 3.8. Find the unique transformation p taking $-i \mapsto \infty$, $i \mapsto 0$, $-3i \mapsto 1$.

The proof above is constructive - it gives us a method for making such a map every time. However in practice it can often be quicker to make it using common sense especially when ∞ is one of the target points. We want $-i \mapsto \infty$, $i \mapsto 0$, so we try

$$p(z) = k\frac{z-i}{z+i} \text{ for some } k \in \mathbb{C}$$

Then $p(-3i) = \frac{-4ki}{-2i} = 2k$, so we take $k = \frac{1}{2}$. Hence $p(z) = \frac{z-i}{2z+2i}$.

An important concept in projective geometry is that of the cross ratio, a quantity that is preserved by projective transformations of $\mathbb{CP}^1$ - the complex projective line. This is a useful tool to help in our understanding of Möbius transformations and we see below in 3.10 this quantity is also invariant under elements of $M\ddot{o}b(\hat{\mathbb{C}})$. We will discover later that this is no coincidence.

Definition 3.9. [2, pp. 37] For distinct $z_1, z_2, z_3, z_4 \in \hat{\mathbb{C}}$ we define the *cross ratio*,

$$[z_1, z_2, z_3, z_4] = \frac{(z_1 - z_4)(z_3 - z_2)}{(z_1 - z_2)(z_3 - z_4)}$$

Remark. The map taking $z \mapsto [z, z_2, z_3, z_4]$ is the function sending z_2, z_3, z_4 to $\infty, 1, 0$ respectively, similar to that which we used in the proof of 3.7.

Proposition 3.10. *Elements of $M\ddot{o}b(\hat{\mathbb{C}})$ preserve the cross ratio. For any Möbius transformation f, $[z_1, z_2, z_3, z_4] = [f(z_1), f(z_2), f(z_3), f(z_4)]$*

Proof. By 3.5, we can decompose f into the composition of maps the types $(T1), (T2), (T3)$, so all we have to do is show that they each preserve the cross ratio:

$$\begin{aligned}
&[az_1, az_2, az_3, az_4] \\
&= \frac{(az_1 - az_4)(az_3 - az_2)}{(az_1 - az_2)(az_3 - az_4)} \\
&= \frac{(z_1 - z_4)(z_3 - z_2)}{(z_1 - z_2)(z_3 - z_4)} \\
&= [z_1, z_2, z_3, z_4]
\end{aligned}$$

$$\begin{aligned}
&[z_1 + b, z_2 + b, z_3 + b, z_4 + b] \\
&= \frac{((z_1 + b) - (z_4 + b))((z_3 + b) - (z_2 + b))}{((z_1 + b) - (z_2 + b))((z_3 + b) - (z_4 + b))} \\
&= \frac{(z_1 - z_4)(z_3 - z_2)}{(z_1 - z_2)(z_3 - z_4)} \\
&= [z_1, z_2, z_3, z_4]
\end{aligned}$$

$$\begin{aligned}
\left[\frac{1}{z_1}, \frac{1}{z_2}, \frac{1}{z_3}, \frac{1}{z_4}\right] &= \frac{(\frac{1}{z_1} - \frac{1}{z_4})(\frac{1}{z_3} - \frac{1}{z_2})}{(\frac{1}{z_1} - \frac{1}{z_2})(\frac{1}{z_3} - \frac{1}{z_4})} \\
&= \frac{\frac{1}{z_4 z_1}(z_4 - z_1)\frac{1}{z_2 z_3}(z_2 - z_3)}{\frac{1}{z_1 z_2}(z_2 - z_1)\frac{1}{z_3 z_4}(z_4 - z_3)} \\
&= \frac{(z_1 - z_4)(z_3 - z_2)}{(z_1 - z_2)(z_3 - z_4)} \\
&= [z_1, z_2, z_3, z_4]
\end{aligned}$$

□

It turns out that the cross ratio is not the only quantity preserved by Möbius transformations.

Proposition 3.11. *Möbius transformations map generalised circles to generalised circles.*

A proof for this proposition can be found in either [2, pp. 26] or in [4, pp. 206].

The general idea is this: by proposition 3.5 we can write any Möbius transformation as the composition of simpler functions. We can see that types (T1) and (T2) will map $\infty \mapsto \infty$, circles and lines in $\mathbb{C}$ map to circles and lines in $\mathbb{C}$ respectively, so preserve generalised circles. Therefore, all that needs to be shown is that the inversion map preserves circles in $\hat{\mathbb{C}}$.

While we have not included the proof, it is still useful to consider what happens to different classes of generalised circle under this map to help our visualisation of this transformation.

There are 4 types of generalised circles:

(1) circles in $\mathbb{C}$ that pass through the origin
(2) circles in $\mathbb{C}$ not passing through the origin
(3) lines[3] that pass through the origin
(4) lines that do not pass through the origin

The inversion sends type (1) $\mapsto$ (4), (4) $\mapsto$ (1), (2) $\mapsto$ (2), (3) $\mapsto$ (3). It is important to note that although this says that circles of type (2) go to circles of type (2), this will almost always not be the same circle.

We can use this to derive an extremely nice (and useful) property of the cross ratio.

Corollary 3.12. $[z_1, z_2, z_3, z_4] \in \mathbb{R} \Leftrightarrow z_1, z_2, z_3, z_4$ *lie on a generalised circle.*

Proof. [2, pp. 38]. Because z_2, z_3, z_4 are distinct $\exists g \in M\ddot{o}b(\hat{\mathbb{C}})$ such that $g(z_2) = \infty$, $g(z_3) = 1$, $g(z_4) = 0$. By 3.10

$$\begin{aligned}
[z_1, z_2, z_3, z_4] &= [g(z_1), g(z_2), g(z_3), g(z_4)] = [g(z_1), \infty, 1, 0] \\
&= \frac{(g(z_1) - 0)(1 - \infty)}{(g(z_1) - \infty)(1 - 0)} = \frac{g(z_1)}{1} \frac{(1 - \infty)}{(g(z_1) - \infty)} \\
&= g(z_1) \frac{(1/\infty) - 1}{(g(z_1)/\infty) - 1} = g(z_1) \frac{-1}{-1} \\
&= g(z_1)
\end{aligned}$$

Hence $[z_1, z_2, z_3, z_4] \in \mathbb{R} \Leftrightarrow g(z_1) \in \mathbb{R}$. We know z_1, z_2, z_3, z_4 lie on a generalised circle if and only if $g(z_1), g(z_2), g(z_3), g(z_4)$ also do, which is the case if and only if $g(z_1) \in \mathbb{R}$, as $g(z_1)$ must be on the line through 0 and 1 which is the real axis. $\square$

This gives us a superb test to see quickly whether or not 4 complex numbers are collinear or concyclic.

Actually, by examining the cross ratio of 4 points there is a stronger result that allows us to say even more about them geometrically (from [6, pp. 155]). Given a complex point p, another way of stating the above corollary, is that p lies on the generalised circle C through q, r, s if and only if $\mathrm{Im}[p, q, r, s] = 0$. It is also further true that if q, r, s produce a positive (resp. negative) orientation[4] on C, that p lies inside C if and only if $\mathrm{Im}[p, q, r, s] < 0$ (resp. $\mathrm{Im}[p, q, r, s] > 0$).

Note that we could alternatively have shown this property of the cross ratio using a geometrical argument, then easily deduced from that 3.11, by using that Möbius transformations preserve the cross ratio.

In [2, pp. 51] is it shown that $M\ddot{o}b(\hat{\mathbb{C}})$ is precisely the set of bijective continuous maps that map circlines to circlines of $\hat{\mathbb{C}}$, and it is shown further that all such maps are conformal (angle preserving). Recall that the stereographic projection also sends circlines to circlines. Using this identification, it is then shown that $M\ddot{o}b(\hat{\mathbb{C}})$ is precisely the set of all bijective conformal maps from the Riemann sphere onto itself.

4 Classification

We will now try and classify this group and one notion that we will be using is that of a fixed point. We define a *fixed point* of f, as $z \in \hat{\mathbb{C}}$ such that $f(z) = z$.

Theorem 4.1. *Let f(z) be a Möbius transformation which has 3 distinct fixed points. Then f is the identity.*

[3] Remember that the whole generalised circle is the line, union the point at infinity.
[4] This means if you move along the path q, r, s then the interior of the circle is to your left (resp. right).

Proof. [2, pp. 30] Take $m \in M\ddot{o}b(\hat{\mathbb{C}})$. Assume that $m \neq id$. $m(z) = \frac{az+b}{cz+d}$ for some $a, b, c, d \in \mathbb{C}$. $m(\infty) = \frac{a}{c}$ so $m(\infty) = \infty \Leftrightarrow c = 0$.

Case 1. $c = 0$

Then $m(z) = \frac{a}{d}z + \frac{b}{d}$ ($d \neq 0$ as $ad - bc \neq 0$), so the fixed points in $\mathbb{C}$ are the solutions of the equation $m(z) = \frac{a}{d}z + \frac{b}{d} = z$. If $\frac{a}{d} = 1$, then because m is not the identity $b \neq 0$. Then there are no solutions, thus no fixed points in $\mathbb{C}$.

Otherwise, $\frac{a}{d} \neq 1$ so So $z = \frac{b}{d-a}$ is the only fixed point in $\mathbb{C}$. Therefore, if $c = 0$, as ∞ is also a fixed point, m has either 1 or 2 fixed points.

Case 2. $c \neq 0$

$m(\infty) \neq \infty$, so the fixed points are the solution that lie in $\mathbb{C}$ of the equation $m(z) = \frac{az+b}{cz+d} = z$, which are the roots of the quadratic $cz^2 + (d-a)z - b = 0$. Therefore, if $c \neq 0$, m has 1 or 2 fixed points.

So the identity is the only Möbius transformations having more then 2 fixed points. □

Therefore, if we now assume that f is not the identity map, then we can deduce the following from the details of the preceding proof.

Corollary 4.2. *If $f \in M\ddot{o}b(\hat{\mathbb{C}})$, and $f \neq id$, then f has either 1 or 2 fixed points.*

Finding the fixed points is relatively easy. You just set $z = f(z)$ and solve for z.

Example 4.3. Find the fixed points of $m_1(z) = \frac{2z+i}{iz}$, and of $m_2(z) = \frac{z-2}{z-1}$.

Fixed points of m_1 satisfy $z_* = \frac{2z_*+i}{iz_*}$, so $iz_*^2 - 2z_* - i = 0$. Hence

$$z_* = \frac{2 \pm \sqrt{4 + 4i^2}}{2i} = \frac{2 \pm \sqrt{4 - 4}}{2i} = \frac{1}{i} = -i$$

Fixed points of m_2 satisfy $z_* = \frac{z_*-2}{z_*-1}$, so $iz_*^2 - 2z_* + 2 = 0$. Hence

$$z_\pm = \frac{2 \pm \sqrt{4-8}}{2} = \frac{2 \pm 2i}{2i} = 1 \pm i$$

4.1 Conjugate Transformations

Definition 4.4. Möbius tranformations, f, g are said to be *conjugate* if $\exists\, p \in M\ddot{o}b(\hat{\mathbb{C}})$ such that $f = p \circ g \circ p^{-1}$. If so we write $f \sim g$.

You should notice that conjugation (as it's name suggests) is nothing but the conjugation group action of $M\ddot{o}b(\hat{\mathbb{C}})$ on itself, with the conjugacy class of an element being the orbit of that element. Therefore, we know that conjugation is an equivalence relation, so partitions $M\ddot{o}b(\hat{\mathbb{C}})$.

Furthermore, the identity element is always the only element of it's orbit, so we know the conjugacy class of the identity map is the set just containing itself.

In $PSL(2, \mathbb{C})$, the corresponding relation[5] is similarity of matrices up to sign. We will therefore from now on say $[A]$ is similar to $[B] \Leftrightarrow A$ is similar to B or A is similar to $-B$. Recall that 2 matrices being similar means they represent the same map with respect to different coordinates.

Geometrically, if f, g are conjugate, then the effect of f on $\mathbb{C}$ is the same as the effect of g on $p(\mathbb{C}) = \mathbb{C}$. So conjugation represents a change of coordinates[6].

Lemma 4.5. *If $f \sim g$, so $f = p \circ g \circ p^{-1}$, then f and g have the same number of fixed points.*

Proof. If $f = id$, then $g = id$, so f and g being the same transformation have the same number of fixed points.

If $f \neq id$, then $g \neq id$ and both f and g have 1 or 2 fixed points. Call these z_i and let N_f, N_g be the number of such points of f and g respectively. Then $z_i = f(z_i)$ so $p^{-1}(z_i) = p^{-1}(f(z_i)) = g(p^{-1}(z_i))$, hence $p^{-1}(z_i)$ is a fixed point of g. As Möbius transformations are injective $p^{-1}(z_i) \neq p^{-1}(z_j)$ for $i \neq j$, hence $N_f \leqslant N_g$. Congruence is symmetric, so by the same reasoning, $N_g \leqslant N_f$. So $N_f = N_g$. □

[5]Again this is just $PSL(2, \mathbb{C})$ acting on itself under the conjugation action.

[6]This is not typically linear, as shall be explained shortly.

4.2 Standard Form

For this section, mostly taken from [2, pp. 40], we assume $m \neq id$. Depending on the number of fixed points, we now try to find class representatives for each conjugacy class.

Suppose m has 1 fixed point z^* in $\hat{\mathbb{C}}$. Let w be any element of $\hat{\mathbb{C}} \setminus \{z^*\}$. z^* is the only fixed point, so $(z^*, w, m(w))$ is a triple of distinct points in $\hat{\mathbb{C}}$ so there exists[7] $p \in M\ddot{o}b(\hat{\mathbb{C}})$ that takes this triple to $(\infty, 0, 1)$. So $(p \circ m \circ p^{-1})(\infty) = p(m(z^*)) = p(z^*) = \infty$. Hence ∞ is a fixed point of this composition.

Therefore we can write $(p \circ m \circ p^{-1})(z) = az + b$ for some $a \neq 0$. (Previously we have shown $m(\infty) = \infty \Leftrightarrow c = 0$). From the above Lemma, $p \circ m \circ p^{-1}$ also must have one fixed point which we have shown is ∞. So there are no solutions in $\mathbb{C}$ to the equation $(p \circ m \circ p^{-1})(z) = az + b = z$ which means that $a = 1$. Additionally, $(p \circ m \circ p^{-1})(0) = p(m(y)) = 1$, then $b = 1$.

Therefore, Möbius transformations m with exactly one fixed point are conjugate to $n_1 : z \mapsto z + 1$. We call such transformations *parabolic*, and n_1 it's standard form.

Example 4.6. Taking $m_1(z) = \frac{2z+i}{iz}$, we showed earlier that this has one fixed point of $-i$. Using the constructive method of the proof, take $y = i$, so $m_1(y) = -3i$. So we want find the transformation taking $-i \mapsto \infty$, $i \mapsto 0$, $-3i \mapsto 1$. We already showed in example 3.8 that this is $p(z) = \frac{z-i}{2z+2i}$. Let's make use of the matrix correspondence to verify that $p \circ m \circ p^{-1} = n_1$. Taking the unique matrix (up to sign) to represent each map

$$\begin{pmatrix} 1 & -1 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ i & 0 \end{pmatrix} \left(\frac{1}{4i} \begin{pmatrix} 2i & i \\ -2 & i \end{pmatrix} \right) = \frac{1}{4i} \begin{pmatrix} 1 & -i \\ 2 & 2i \end{pmatrix} \begin{pmatrix} 2i & 3i \\ -2 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

Which indeed corresponds to n_1.

Suppose instead that m has 2 fixed points z_1^*, z_2^* in $\hat{\mathbb{C}}$. Take $q \in M\ddot{o}b(\hat{\mathbb{C}})$ satisfying $q(z_1^*) = 0, q(z_2^*) = \infty$. Because of the way we chose q, $(q \circ m \circ q^{-1})(\infty) = q(m(z_2^*)) = q(z_2^*) = \infty$ so $(q \circ m \circ q^{-1})(z) = az + b$ for some $a \neq 0$ as before, and $(q \circ m \circ q^{-1})(0) = q(m(z_1^*)) = q(z_1^*) = 0$, so $b = 0$. Also $a \neq 1$ because id is in a conjugacy class of it's own.

Therefore, Möbius transformations m with exactly 2 fixed points are conjugate to $n_2 : z \mapsto az$ for some $a \in \mathbb{C} \setminus \{0, 1\}$. We call a the *multiplier of* m, and n_2 it's standard form.

Example 4.7. We showed earlier that $m_2(z) = \frac{z-2}{z-1}$ has 2 fixed points of $1 \pm i$. $q(z) = \frac{z-(1-i)}{z-(1+i)}$ satisfies $q(1-i) = 0$, $q(1+i) = \infty$. Again we can use the matrix correspondence to verify that $q \circ m \circ q^{-1} = n_2$. Taking the unique matrix (up to sign) to represent each map you can verify yourself that

$$\begin{pmatrix} 1 & -(1-i) \\ 1 & -(1+i) \end{pmatrix} \begin{pmatrix} 1 & -2 \\ 1 & -1 \end{pmatrix} \left(\frac{1}{2i} \begin{pmatrix} i+i & i-1 \\ 1 & -1 \end{pmatrix} \right) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

So this corresponds to $n_2 : z \mapsto \frac{iz}{-i} = -z$. Hence the multiplier is -1.

Like before (with p), this q is not unique. However we see below that this does not really matter, from this lemma from [2, pp. 41].

Lemma 4.8. *Suppose $m \in M\ddot{o}b(\hat{\mathbb{C}})$ has 2 fixed points $x, y \in \hat{\mathbb{C}}$, and that $q_1, q_2 \in M\ddot{o}b(\hat{\mathbb{C}})$ satisfy $q_1(x) = q_2(x) = 0$, $q_1(y) = q_2(y) = \infty$ then $q_1 \circ m \circ q_1^{-1} = q_2 \circ m \circ q_2^{-1}$. (So the multiplier a is the same.)*

Furthermore, if $q \in M\ddot{o}b(\hat{\mathbb{C}})$ satisfies $q(x) = \infty$, $q(y) = 0$ then the multiplier of $q \circ m \circ q^{-1}$ is $\frac{1}{a}$

Corollary 4.9. *The multiplier is defined up to it's inverse. Furthermore, if $m(z) = az$, $T(z) = \frac{1}{z}$, then $T \circ m \circ T^{-1}(z) = \frac{1}{a}z = m^{-1}(z)$*

This tells us that the standard form for a non parabolic transformation (i.e. any transformation with 2 fixed points) is unique up to the multiplicative inverse of the multiplier - and furthermore, that these sit in the same conjugacy class, in particular being conjugate by the inversion map.

[7] Note that even though a Möbius transformations uniquely exists taking $(z^*, w, m(w)) \mapsto (\infty, 0, 1)$, p is not unique. This arises from the choice we had for w.

We can classify further depending on the multiplier. Recall $a \in \mathbb{C} \setminus \{0, 1\}$.

If $|a| = 1$, for some $\theta \in (0, 2\pi)$ we can write $a = e^{i\theta}$, so the transformation is a rotation clockwise by angle θ. We call m *elliptic* and say that $q \circ m \circ q^{-1} = e^{i\theta} z$ is it's standard form.

The final case is if $|a| \neq 1$. So $a = re^{i\theta}$ for $r \in \mathbb{R} \setminus \{0, 1\}$, $\theta \in [0, 2\pi)$. m is said to be *loxodromic* with $q \circ m \circ q^{-1} = re^{i\theta} z$, $\theta \neq 0$ is it's standard form. In the special case where $\theta = 0$ (a pure dilation) then we say m is *hyperbolic* - having standard form $q \circ m \circ q^{-1} = rz$.

Observe that if m is loxodromic, then it is the composition of dilation by r and a rotation by θ, both about 0, performed in either order; hence it's standard form is just the composition of hyperbolic and elliptic transformations.

Example 4.10. Taking $m_2(z) = \frac{z-2}{z-1}$ as before, because the multiplier is -1, then this is elliptic and m_2 is conjugate to a rotation by π.

Now there is a fast method of determining the standard form of any given Möbius transformation. First we need to introduce the following function

Definition 4.11. For any $m = \frac{az+b}{cz+d} \in M\ddot{o}b(\hat{\mathbb{C}})$ we define $\tau : M\ddot{o}b(\hat{\mathbb{C}}) \to \mathbb{C}$, by $\tau(m) = (a+d)^2$. The respective map for $[A] \leftrightarrow m$ from $PSL(2, \mathbb{C}) \to \mathbb{C}$ is $\text{tr}^2 : [A] \mapsto (a+d)^2$.

The symbol τ is chosen for this function as we are motivated by the trace of a matrix. Note that if we had defined instead $\tau(m) = a + d$, then this would be ill defined, due to the dual representation of each map, however τ is well defined as $((-a) + (-d))^2 = (a+d)^2$.

Using this new function, in [2, pp. 45] you can find the proof for the following:

Proposition 4.12. *Let $m \in M\ddot{o}b(\hat{\mathbb{C}}) \setminus \{id\}$. Then:*

(a) m *is parabolic* $\Leftrightarrow \tau(m) = 4$
(b) m *is elliptic* $\Leftrightarrow \tau(m) \in \mathbb{R}$ *and* $0 \leq \tau(m) < 4$
(c) m *is hyperbolic* $\Leftrightarrow \tau \in \mathbb{R} \setminus (-\infty, 4]$
(d) m *is loxodromic* $\Leftrightarrow \tau \in \mathbb{C} \setminus [0, 4]$

Now we have a very useful tool that enables us to immediately determine the type of a given transformation. We can go further and see that τ determines the conjugacy class entirely.

Lemma 4.13. *If $f, g, p \in M\ddot{o}b(\hat{\mathbb{C}})$, then $\tau(f \circ g) = \tau(g \circ f)$ and $\tau(p \circ f \circ p^{-1}) = \tau(f)$.*

Proof. (Sketch)
To see $\tau(f \circ g) = \tau(g \circ f)$, use the correspondence with matrices and that $\text{tr}(AB) = \text{tr}(BA)$ for matrices A, B. Then, $\tau(p \circ f \circ p^{-1}) = \tau(p^{-1} \circ p \circ f) = \tau(f)$ □

Due to the second property in 4.13, it it enough to just consider the values of τ on the standard forms.

Now take $f, g \in M\ddot{o}b(\hat{\mathbb{C}})$, satisfying $\tau(f) = \tau(g)$. First, if $\tau(f) = 4 = \tau(g)$, then f and g are parabolic. So they have the same standard form $n_1 : z \mapsto z + 1$. So $f \sim g$.

Now if $\tau(f) \neq 4$, let α be the multiplier of f, β the multiplier of g. So

$$\tau(f) = \tau(g) \Rightarrow (\sqrt{\alpha} + \frac{1}{\sqrt{\alpha}})^2 = (\sqrt{\beta} + \frac{1}{\sqrt{\beta}})^2 \Rightarrow \alpha + \frac{1}{\alpha} + 2 = \beta + \frac{1}{\beta} + 2 \Rightarrow \alpha + \frac{1}{\alpha} = \beta + \frac{1}{\beta}$$

which has solutions $\alpha = \beta$ and $\alpha = \frac{1}{\beta}$. Then by corollary 4.9, we can see that the standard forms are either equal (so trivially conjugate) or conjugate by the inversion map, so (as conjugation is an equivalence relation), $f \sim g$.

Coupling this with 4.13, we get that $f \sim g \Leftrightarrow \tau(f) = \tau(g)$. So the multiplier is completely determined by τ and vice versa. We state this as a proposition:

Proposition 4.14. *$f \sim g \Leftrightarrow \tau(f) = \tau(g)$. In terms of matrices, given A, B - matrices corresponding to f, g resp. , then $[A]$ is similar[8] to $[B]$* $\Leftrightarrow \text{tr}(A)^2 = \text{tr}(B)^2$

[8]We defined what similarity of elements of $PSL(2\mathbb{C})$ means at the start of 4.1.

4.3 Back to Matrices

It turns out we can exploit the relationship with matrices further and grow to understand it more in the process. We seemed to have stumbled upon it purely by chance! For example, what do they represent? We are used to 2 by 2 matrices acting as linear transformations of $\mathbb{R}^2$ or $\mathbb{C}$ with real entries. However we can quickly see this is not the case here; our matrices have complex entries, and consider $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, the matrix for the inversion, which is not a linear transformation of $\mathbb{C}$.

You may further ask *why* this nice correspondence exists. To answer that we must briefly detour to the world of projective geometry as alluded to earlier, with this section based roughly on [6, pp. 157].

We begin by describing the complex plane in a different and completely new kind of coordinate system - instead of writing $z = x + iy$ in terms of 2 *real* numbers, let us write $z = \frac{\alpha}{\beta}$, the ratio of 2 *complex* numbers. As shorthand we write $[\alpha, \beta]$ and call these the projective coordinates of z. To ensure that this ratio is well defined we do not allow $[\alpha, \beta] = [0, 0]$. This is very different from the way are used to writing complex numbers - particularly as the coordinates are not unique; $[\alpha, \beta] = [k\alpha, k\beta]$ for any $k \in \mathbb{C} \setminus \{0\}$. To all pairs $[\alpha, \beta]$ with $\alpha \in \mathbb{C}$, $\beta \in \mathbb{C} \setminus \{0\}$ there exists uniquely one point in $\mathbb{C}$. Note this excludes the pair $[\alpha, 0]$ (when $\beta = 0$) as this corresponds to $z = \frac{\alpha}{0}$. However, working instead in $\hat{\mathbb{C}}$ we identify $[\alpha, 0]$ with ∞. Thus in this coordinate system the point ∞ is no longer exceptional, it is just a complex pair (i.e. an element of $\mathbb{C}^2$) just like any other. This is much like when we identified ∞ with N on the Riemann Sphere, but this time from an algebraic point of view, rather then a geometrical one.

We can think of 2 by 2 complex matrices as linear transformations of $\mathbb{C}^2$, just in the same way that 2 by 2 real matrices are linear transformations of $\mathbb{R}^2$. They map

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} a\alpha + b\beta \\ c\alpha + d\beta \end{pmatrix}$$

But now we have a new way to view this pair, from the perspective of projective coordinates! If we think of $[\alpha, \beta]$ as representing $z = \frac{\alpha}{\beta} \in \hat{\mathbb{C}}$, and the above linear transformation is interpreted as the non linear transformation of $\hat{\mathbb{C}}$, taking

$$z = \frac{\alpha}{\beta} \mapsto \frac{a\alpha + b\beta}{c\alpha + d\beta} = \frac{a\frac{\alpha}{\beta} + b}{c\frac{\alpha}{\beta} + d} = \frac{az + b}{cz + d}$$

Which is exactly the corresponding Möbius transformation!

Now we understand better why Möbius transformations behave so much like linear transformations and have such nice matrix correspondence; they are precisely linear transformations, but on the projective coordinates in $\mathbb{C}^2$, not directly on $\mathbb{C}$.

Now let's put this beautiful new discovery to good use. Without giving a formal proof, by considering what it means to be an eigenvector, and remembering that in projective coordinates $[\alpha, \beta] = \lambda[\alpha, \beta]$ for any non-zero $\lambda \in \mathbb{C}$, we see that

Proposition 4.15. *$z = \frac{\alpha}{\beta}$ is a fixed point of $m \Leftrightarrow \left(\begin{smallmatrix} \alpha \\ \beta \end{smallmatrix}\right)$ is an eigenvector of the corresponding matrix*[9].

Example 4.16. Let $f(z) = 4z + 8i$. Then in $PSL(2, \mathbb{C})$ this is[10] $[A]$, where $A = \begin{pmatrix} 2 & 4i \\ 0 & \frac{1}{2} \end{pmatrix}$.
A has eigenvalues satisfying:

$$\lambda^2 - \lambda(a + d) + 1 = \lambda^2 - \frac{5}{2}\lambda + 1 = 0$$

So they are given by the equation

$$\lambda_\pm = \frac{\frac{5}{2} \pm \sqrt{(\frac{5}{2})^2 - 4}}{2} = \frac{\frac{5}{2} \pm \frac{3}{2}}{2} \qquad \text{so} \qquad \lambda_+ = 2, \quad \lambda_- = \frac{1}{2}$$

Solving $A\mathbf{v} = \lambda_\pm \mathbf{v}$, we find that λ_+ and λ_- have associated eigenvectors $\mathbf{v}_+ = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$, $\mathbf{v}_- = \left(\begin{smallmatrix} -8i \\ 3 \end{smallmatrix}\right)$.

Then the fixed points are $\frac{1}{0} = \infty$, and $\frac{-8i}{3} = -\frac{8}{3}i$.

[9] Remember this is up to sign, but fine, as $A\mathbf{v} = \lambda\mathbf{v} \Leftrightarrow (-A)\mathbf{v} = (-\lambda)\mathbf{v}$

[10] After we normalise the transformation.

By looking at fixed points from this point of view there is really no difference between a fixed point in $\mathbb{C}$ or at ∞, just as there is no real difference between the north pole and any other point on the sphere.

Can we gain extra information from this alternative formulation of a fixed point? It turns out that we can, by investigating the value of the eigenvalue.

First we define the matrices of the standard forms.

$$M_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad M_a = \begin{pmatrix} \sqrt{a} & 0 \\ 0 & \frac{1}{\sqrt{a}} \end{pmatrix} \text{ for } a \in \mathbb{C} \setminus \{0, 1\}$$

So the standard form of a parabolic transformation is $[M_1]$ in $PSL(2, \mathbb{C})$, and any other standard form, with multiplier a, is $[M_a]$ in $PSL(2, \mathbb{C})$.

Now we know given $[A] \in PSL(2, \mathbb{C}) \setminus \{[I_2]\}$, that $[A]$ is similar to either $[M_1]$ or $[M_a]$ for some $a \in \mathbb{C} \setminus \{0, 1\}$. Similar matrices have the same eigenvalues, and here with our slightly altered definition the same still holds, but as usual up to the sign of the eigenvalue (λ is an eigenvalue of A $\Leftrightarrow -\lambda$ is an eigenvalue of $-A$).

Therefore, we can conclude that the eigenvalues tell us precisely the standard form, and give us the relationship that the multiplier of the transformation represented by the matrix is $a = \lambda^2$.

Example 4.17. Consider again f as in 4.16 above. We have already calculated the eigenvalues, and so can deduce that the multiplier is 4 if we use λ_+, and $\frac{1}{4}$ if we use λ_-. This choice is not an issue as we know the multiplier is only defined up to it's inverse[11]. So this transformation is conjugate to the standard form $n : z \mapsto 4z$, and is hyperbolic.

We can thus summarise our results as follows:

Type	**Fixed Points**	τ	**Multiplier**	**Class Representative**
Parabolic	1	$\tau = 4$	-	$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$
Elliptic	2	$\tau \in [0, 4)$	$e^{i\theta}$ s.t. $\theta \in (0, 2\pi)$	$\begin{pmatrix} e^{\frac{i\theta}{2}} & 0 \\ 0 & e^{-\frac{i\theta}{2}} \end{pmatrix}$
Hyperbolic	2	$\tau \in \mathbb{R} \setminus (-\infty, 4]$	e^{θ} s.t. $\theta \in \mathbb{R} \setminus \{0\}$	$\begin{pmatrix} e^{\frac{\theta}{2}} & 0 \\ 0 & e^{-\frac{\theta}{2}} \end{pmatrix}$
Loxodromic	2	$\tau \in \mathbb{C} \setminus [0, 4]$	a s.t. $\lvert a\rvert \neq 1$	$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-}1 \end{pmatrix}$ s.t. $\lambda^2 = a$

Some texts go even further and define spherical as a subset of elliptical, being those with $\tau = 0$. These the transformations with multiplier -1, so are conjugate to the standard form sending $z \mapsto -z$, a reflection in the imaginary axis. Note that $z \mapsto \frac{1}{z}$ is also in this conjugacy class (has $\tau = 0$).

4.4 Normal Form

We alternatively can think about Möbius transformations as iterative maps in terms of their fixed points, as is done in [6, pp. 169].

Let $m \in M\ddot{o}b(\hat{\mathbb{C}})$ be a non identity map have 2 distinct fixed points, z_+, z_- (m is any non parabolic transformation). Now take $q = \frac{z - z_+}{z - z_-}$ as in section 4.2, a map satisfying $q(z_+) = 0$, $q(z_-) = \infty$. We know

[11] This will always be the case as $\lambda_+\lambda_- = \det(A) = 1$, a property of eigenvalues.

that $q \circ m \circ q^{-1} = n$, where $n : z \mapsto az$ for some multiplier a. Rewriting, as $q \circ m = n \circ q$, and $w = m(z)$ (the image under m) we see:

$$\frac{w - z_+}{w - z_-} = a\frac{z - z_+}{z - z_-}$$

Now call $w^{(k)} = m^k(z)$, the image of z under k applications of m, and z_0 the starting point. Then $m = q^{-1} \circ n \circ q$, $m^k = (q^{-1} \circ n \circ q)^k = q^{-1} \circ n^k \circ q$, so $q \circ m^k = n^k \circ q$. Thus

$$\frac{w^{(k)} - z_+}{w^{(k)} - z_-} = a^k \frac{z_0 - z_+}{z_0 - z_-} = a^k C$$

Where C is a constant.

We can then solve for $w^{(k)}$ to yield:

$$w^{(k)} = \frac{z_+ - a^k C z_-}{1 - a^k C}$$

For $z_0 \neq z_+, z_-$, if $|a| > 1$, then we see that the a^k term dominates, so $w^{(k)} \mapsto z_-$, as $k \mapsto \infty$, and if $|a| < 1$, then $a^k \mapsto 0$ as $k \mapsto \infty$, so $w^{(k)} \mapsto z_+$. Note that if $|a| = 1$, the limit does not exist. This agrees geometrically, as elliptical maps just rotate points around the fixed points. In the cases where $z_0 = z_+$ or $z_0 = z_-$, then we know these stay put as $k \mapsto \infty$, being fixed points.

So for non parabolic transformations, points either flow from one fixed point to the other, or oscillate around them. In particular, for hyperbolic transformations, these points move along circles through the 2 fixed points.

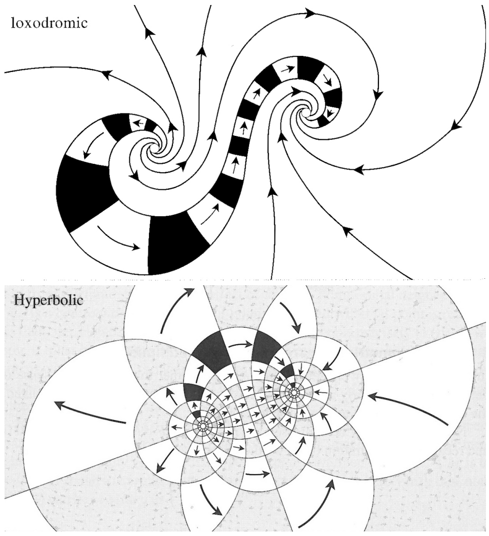

Figure 2: Iterating a loxodromic and hyperbolic transformation

If instead $m(z) = \frac{az+b}{cz+d}$ is parabolic and has one fixed point $z^* = \frac{a-d}{2c} \in \hat{\mathbb{C}}$, then if z_* is finite the normal form is

$$\frac{1}{w - z_*} = \frac{1}{z - z_*} \pm c \qquad \text{so} \qquad \frac{1}{w^{(k)} - z_*} = \frac{1}{z^{(0)} - z_*} \pm kc$$

where sign of c is positive if $a + d = 2$, negative if $a + d = -2$.

If $z_* = \infty$, then we know that $c = 0$, and as $ad - bc = 1$, then $a = d = \pm 1$ so $m(z) = z \pm b$. Hence the normal form is[12]

$$w = z \pm b \qquad \text{so} \qquad w^{(k)} = z^{(0)} \pm kb$$

In both of these cases we can see that $\lim_{k \mapsto \infty} w^{(k)} = z_*$.

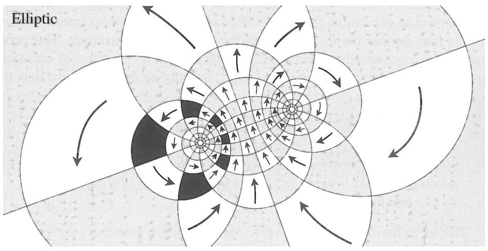

Figure 3: Iterating an elliptic transformation

So for parabolic transformations, all points move towards the fixed point.

4.5 Geometry of the Standard Forms

Finally, we can now visualise what these standard forms look like on the sphere.

For the transformation with 2 fixed points, in all the standard forms that 0 and ∞ are fixed points, which correspond to the north and south pole on $\mathbf{\Sigma}$. Then using the normal form above, we see for each standard form that points move along lines as in [a], [b], and [c] in figure 4. It is interesting that the name loxodromic derives from the fact that such transformations make constant angles with the lines of longitude.

For parabolic transformations, the standard form has only ∞ as a fixed point at the top of the sphere, and all points move towards it, in a fashion exhibited in [d], figure 4.

[12] We also know b $\neq 0$, as $m \neq id$.

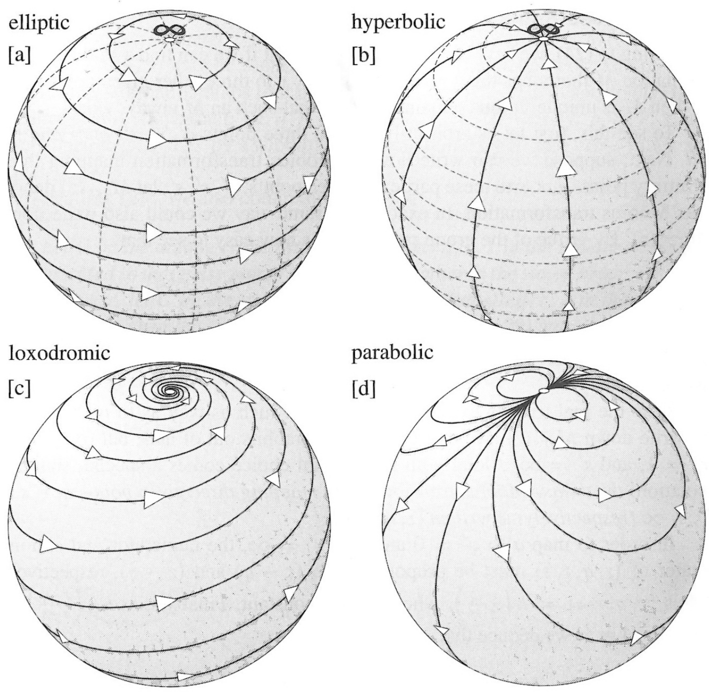

Figure 4: The effect of the standard forms on $\mathbf{\Sigma}$

5 Motions of the Sphere

If you can, cast your mind back to when we described the inversion in terms of a rotation of the Riemann sphere. Here we see that there was nothing special about the inversion - we can do it for any Möbius transformation! Additionally, these are the only maps for which this property holds!

This is shown in [3], where the following elegant result is stated and proven.

Theorem 5.1. *A complex mapping is a Möbius transformation if and only if it can be obtained by stereographic projection of the complex plane onto an admissible sphere in* $\mathbb{R}^3$*, followed by a rigid motion of the sphere in* $\mathbb{R}^3$ *which maps it to another admissible sphere, followed by stereographic projection back to the plane.*

An sphere in $\mathbb{R}^3$ is *admissible* if it's north pole, N, lies above the xy-plane. For any admissible sphere, a stereographic projection can be defined as before: for each point P on the sphere, draw the line through N and P and map this to the intersection of this line with the xy plane. So again N is identified with ∞, and the remainder of the sphere with $\mathbb{C}$. The proof runs as follows:

Proof. (Sketch) We use the fact (similar to 3.5) that any Möbius transformation f can be built from (in this order), a translation, inversion, dilation, rotation, and a final translation: we can find α, $\beta \in \mathbb{C}$, ρ, $\theta \in \mathbb{R}$ such that

$$f(z) = \frac{\rho r^{i\theta}}{z+\alpha} + \beta \tag{1}$$

Therefore, we just have to show that for each of these types of map there exists an admissible sphere S and rigid motion T, such that $S' = TS$ is also admissible and we can write each as $P_{S'} \circ T \circ P_S$.

For any translation, we can choose S as any admissible sphere, and T to be the same translation extended to $\mathbb{R}^3$. S' will always be admissible as there is no change in vertical coordinates.

For the other 3 maps, let S be the unit sphere. T for the rotation is also just the rotation extended to $\mathbb{R}^3$. The dilation is more interesting, and we take the corresponding motion as a translation vertically by $\rho - 1$. To produce the inversion, as we showed earlier we take T to be the rotation around the real axis by an angle of π.

Then to write (1) in the form $f = P_{S'} \circ T \circ P_S$, we take S as the unit sphere with centre $-\alpha$, T to be the composition of a translation by α, a rotation by π about the real axis, a rotation by θ about the axis orthogonal to the xy-plane, a translation upwards by $\rho - 1$, and finally a translation by β. □

A beautiful visual representation of this result is shown in the video *Möbius Transformations Revealed*[13], which was created alongside [3] by the authors, bringing this theorem to life.

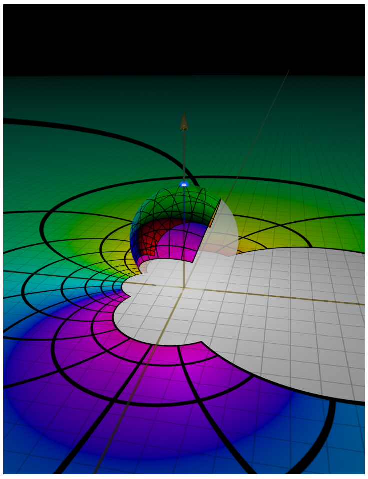

Figure 5: This result visualised in *Möbius Transformations Revealed*

We can see immediately that for a given Möbius transformation, the admissible sphere, S, and rigid motion T, are not necessarily unique. For instance take $f = id$. Then we can let $T = I$, then for any admissible sphere S, $P_{I(S)} \circ I \circ P_S = f$.

However, if we first fix the admissible sphere S, then we get uniqueness of the rigid motion T.

Theorem 5.2. *Let f be a Möbius transformation. For any admissible sphere S, there exists a unique rigid motion T such that $P_{T(S)} \circ T \circ P_S = f$.*

This can be found in [8] and is the main result of the paper. We can still ask more questions though. For example, one open question is given a Möbius transformation, what is the relationship, if any, between the choice of the sphere and T?

6 Further Reading

To whet your appetite for further exploration of these wonderful maps, here are a few amazing results with directions to where you can find more information.

From [6, pp. 123] we see that the complex mappings that correspond to Lorentz transformations of space-time are exactly the Möbius transformations, and conversely, that each Möbius transformations yields a unique Lorentz transformation.

Furthermore, the group properties of $M\ddot{o}b(\hat{\mathbb{C}})$ are extensive: for example, $SO(3)$ - the group of isometries of the sphere, is isomorphic to a subgroup of $M\ddot{o}b(\hat{\mathbb{C}})$ (see [1, pp. 30]) , and the subgroup of Möbius transformations with real coefficients are precisely the motions of 2 dimensional hyperbolic space [6, pp. 313].

References

[1] The geometry of möbius transformations. `http://johno.dk/mathematics/moebius.pdf`. Accessed: 07-04-2018.

[2] James W Anderson. *Hyperbolic Geometry*. Springer Science & Business Media, 2006.

[3] Douglas N. Arnold and Jonathan Rogness. Möbius transformations revealed. *Notices of the American Mathematical Society*, Volume 55:pp.1226–1231, 2008.

[4] John M Howie. *Complex Analysis*. Springer Science & Business Media, 2003.

[5] Kunihiko Kodaira. *Complex analysis*, volume 107. Cambridge University Press, 2007.

[6] Tristan Needham. *Visual complex analysis*. Oxford University Press, 1998.

[7] Hilary A Priestley. *Introduction to Complex Analysis*. OUP Oxford, 1st edition, 1989.

[8] Rob Siliciano. Constructing möbius transformations with spheres. *Rose- Hulman Undergraduate Mathematics Journal*, pages 116–124, Fall 2012.

All images were taken from [6], with the sole exception of figure 5, from the link below.

[13] `http://www-users.math.umn.edu/~arnold/moebius/`